

# Asymptotics of a Nonlinear Fibonacci Recurrence<sup>†</sup>

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## Abstract

We show that the family of sequences defined by

$$h_{n+2} = h_{n+1} + (h_n)^m \text{ where } m > 1, h_0 = 0, h_1 = 1 \quad (1)$$

exhibits superexponential asymptotic behavior. That is, for large  $k$  we have  $h_{2k} \approx (A_m)^{\sqrt{m^{2k}}}$  and  $h_{2k+1} \approx (B_m)^{\sqrt{m^{2k+1}}}$  where  $A_m$  and  $B_m$  are functions of  $m$ .

**Introduction:** It has long been known that the standard Fibonacci numbers satisfy exponential asymptotics. More specifically, if  $F_n$  represents the  $n^{\text{th}}$  Fibonacci number, then for large  $n$  we have  $F_n \approx \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n$ . Furthermore, many variations of the Fibonacci numbers have been studied. For example, [1] studies the  $m = 2$  case of (1) and finds that  $h_{2n} \approx A^{(\sqrt{2})^{2n}}$  and  $h_{2n+1} \approx B^{(\sqrt{2})^{2n+1}}$  for large  $n$ , where  $A = 1.436 \pm .001$  and  $B = 1.451 \pm .001$ . This article is an extension of the methods used in that paper. Here, we prove that their techniques are not just unique to the  $m = 2$  case, rather, they apply for all  $m > 1$ .

**Lemma 1** *If  $n \geq 1$  then  $0 < h_{n+1} \leq h_{n+1} \leq 2(h_n)^m$ .*

*Proof.* Since  $h_n$  is certainly increasing, for  $n \geq 1$  we have  $h_{n+1} = h_n + (h_{n-1})^m \leq h_n + (h_n)^m$  and since  $m > 1$  then  $h_{n+1} \leq (h_n)^m + (h_n)^m = 2(h_n)^m$  ■

For further investigation of the sequence we write

$$h_{n+2} = (h_n)^m \left( 1 + \frac{h_{n+1}}{(h_n)^m} \right), n > 0$$

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and define a new sequence  $\alpha_n = 1 + \frac{h_{n+1}}{(h_n)^m}$  for  $n > 0$ . Then for any  $n > 0$  we have

$$\begin{aligned} h_{n+2} &= (h_{n-2})^{m^2} (\alpha_{n-2})^{m^1} \alpha_n \\ &= (h_{n-4})^{m^3} (\alpha_{n-4})^{m^2} (\alpha_{n-2})^{m^1} \alpha_n \end{aligned}$$

and so on. So, if  $n$  is even, it follows that

$$h_{2k} = (1)^{m^{k-1}} (\alpha_2)^{m^{k-2}} (\alpha_4)^{m^{k-3}} \dots \alpha_{2k}.$$

Which leads to the natural definition  $\alpha_0 = 1$  and allows us to write

$$\begin{aligned} h_{2k} &= \prod_{j=0}^{k-1} (\alpha_{2j})^{m^{k-j-1}} \\ &= \exp \left( \sum_{j=0}^{k-1} \log(\alpha_{2j})^{m^{k-j-1}} \right) \\ &= \exp \left( m \sum_{j=0}^{k-1} \frac{\log \alpha_{2j}}{m^{j+1}} \right) \\ &= \exp \left( \sqrt{m}^{2k} \sum_{j=0}^{k-1} \frac{\log \alpha_{2j}}{m^{j+1}} \right) \\ &= \left( \exp \left( \sum_{j=0}^{k-1} \frac{\log \alpha_{2j}}{m^{j+1}} \right) \right)^{\sqrt{m}^{2k}}. \end{aligned} \tag{2}$$

Next, we define the *growth constant*  $A_m$  by

$$\log A_m = \sum_{j=0}^{\infty} \frac{\log \alpha_{2j}}{m^{j+1}} \tag{3}$$

Thus, it seems plausible to expect that for large enough  $k$  we might have  $h_{2k} \approx (A_m)^{\sqrt{m}^{2k}}$ . Next, proceeding as before but treating odd subscripts instead of even shows that

$$h_{2k+1} = \left( \exp \left( \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} \frac{\log \alpha_{2j+1}}{m^{j+1}} \right) \right)^{\sqrt{m}^{2k+1}} \tag{4}$$

and leads to defining the *growth constant*  $B_m$  by

$$\log B_m = \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} \frac{\log \alpha_{2j+1}}{m^{j+1}}. \tag{5}$$

Similar to our expectation about  $A_m$ , definition (5) may lead to the suspicion that  $h_{2k+1} \approx (B_m)^{\sqrt{m}^{2k+1}}$  for large  $k$ .

It is worthwhile to know whether the series (3) and (5) actually make sense, that is, that they even converge. Also, it will be important for later purposes to obtain an error estimate when approximating  $A_m$  and  $B_m$  through only finitely many terms.

**Lemma 2** *The series that define  $\log A_m$  and  $\log B_m$  converge and all partial sums of the first  $N$  terms are within  $\frac{\log 3}{m^{N+1}(m-1)}$  of the actual sums.*

*Proof.* From lemma 1 we have  $0 < h_{n+1} \leq 2(h_n)^m$  for  $n \geq 1$  so, by definition of  $\alpha_n$ , we have  $1 \leq \alpha_n = 1 + \frac{h_{n+1}}{(h_n)^m} \leq 1 + \frac{2(h_n)^m}{(h_n)^m} = 3$ . Hence,  $0 \leq \log \alpha_n \leq \log 3$  for all  $n$ . Therefore,

$$\log A_m = \sum_{j=0}^{\infty} \frac{\log \alpha_{2j}}{m^{j+1}} \leq \sum_{j=0}^{\infty} \frac{\log 3}{m^{j+1}} \quad (6)$$

and

$$\log B_m = \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} \frac{\log \alpha_{2j+1}}{m^{j+1}} \leq \sum_{j=0}^{\infty} \frac{\log 3}{m^{j+1}}. \quad (7)$$

Since  $m > 1$ , this shows that the series defining  $\log A_m$  and  $\log B_m$  (which both have non-negative terms) are bounded by convergent geometric series. Therefore, they must converge.

Using the bound by the geometric series, we may make the following bounds on the greatest possible error in approximating  $A_m$  and  $B_m$  using  $N$  terms:

$$\begin{aligned} |\text{error}| &\leq \sum_{j=N+1}^{\infty} \frac{\log 3}{m^{j+1}} \\ &= \frac{\log 3}{m^{N+2}} \left( \frac{1}{1 - \frac{1}{m}} \right) \\ &= \frac{\log 3}{m^{N+2}} \frac{m}{m-1} \\ &= \frac{\log 3}{m^{N+1}(m-1)} \blacksquare \end{aligned}$$

We note that the preceding actually gives the following:

**Theorem 1** *The growth constants are continuous functions of  $m$  for  $m \in (1, \infty)$ .*

*Proof.* This follows essentially immediately from the previous lemma: the partial sums (which are continuous functions of  $m$ ) certainly converge uniformly on  $(1 + \delta, \infty)$  for any  $\delta > 0$ . The theorem follows.  $\blacksquare$

It is of possible interest to ask what happens to the growth constants  $A_m$  and  $B_m$  as  $m$  gets large.

**Theorem 2** *The growth constants  $A_m$  and  $B_m$  both  $\rightarrow 1$  as  $m \rightarrow \infty$ .*

*Proof.* From equations (6) and (7), we may bound the series that define  $\log A_m$  and  $\log B_m$  as follows

$$\begin{aligned} \log A_m, \log B_m &\leq \sum_{j=0}^{\infty} \frac{\log 3}{m^{j+1}} \\ &= \frac{\log 3}{m} \left( \frac{1}{1 - \frac{1}{m}} \right) \\ &= \frac{\log 3}{m} \frac{m}{m-1} \\ &= \frac{\log 3}{m-1}. \end{aligned}$$

Since  $\frac{\log 3}{m-1}$  certainly  $\rightarrow 0$  as  $m \rightarrow \infty$ , the theorem is proved. ■

With the aid of the following lemma, we will justify our claims about the asymptotics.

**Lemma 3** *For  $n \geq 3$  we have  $\frac{h_{n-1}}{h_n} \leq \frac{2}{3}$ .*

*Proof.* By lemma 1, we have for  $n \geq 1$ ,  $h_{n+1} \leq 2(h_n)^m$ . So for  $n \geq 3$  we have  $h_{n-1} \leq 2(h_{n-2})^m$ . Dividing by 2 and adding  $h_{n-1}$  to both sides gives

$$0 < \frac{3}{2}h_{n-1} \leq h_{n-1} + (h_{n-2})^m = h_n.$$

Taking reciprocals yields

$$0 < \frac{1}{h_n} \leq \frac{2}{3} \frac{1}{h_{n-1}}.$$

Finally, multiplication by  $h_{n-1}$  gives the desired result. ■

**Theorem 3** *The sequence  $\{h_n\}_{n=0}^{\infty}$  satisfies superexponential asymptotics. That is, for large  $n$ ,  $h_{2n} \approx (A_m)^{\sqrt{m^{2n}}}$  and  $h_{2n+1} \approx (B_m)^{\sqrt{m^{2n+1}}}$ . Which, stated more precisely, means*

$$\lim_{n \rightarrow \infty} \frac{2n}{(A_m)^{\sqrt{m^{2n}}}} = \lim_{n \rightarrow \infty} \frac{h_{2n+1}}{(B_m)^{\sqrt{m^{2n+1}}}} = 1.$$

*Proof.* First consider the limit involving the even subscripts:

$$\begin{aligned} \frac{2n}{(A_m)\sqrt{m^{2n}}} &= \frac{\exp\left(m^n \sum_{j=0}^{n-1} \frac{\log \alpha_{2j}}{m^{j+1}}\right)}{\exp\left(m^n \sum_{j=0}^{\infty} \frac{\log \alpha_{2j}}{m^{j+1}}\right)} \\ &= \exp\left(-\sum_{j=n}^{\infty} m^{n-j-1} \log \alpha_{2j}\right). \end{aligned}$$

*Note:* In order to justify the conjecture, it is necessary to show that

$$\sum_{j=n}^{\infty} m^{n-j-1} \log \alpha_{2j} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We proceed. First, since  $\log(1+x) \leq x$  for  $x > 0$ , then by definition of  $\alpha_n$  we have  $0 \leq \log \alpha_{2j} \leq \frac{h_{2j+1}}{(h_{2j})^m}$ . So, we get

$$\sum_{j=n}^{\infty} m^{n-j-1} \log \alpha_{2j} \leq m^n \sum_{j=n}^{\infty} \frac{1}{m^{j+1}} \frac{h_{2j+1}}{(h_{2j})^m}.$$

By lemma 3, we have  $\frac{h_{n-1}}{h_n} < \frac{2}{3}$  for  $n \geq 3$ . So, since  $m > 1$ , it follows that  $(\frac{h_{n-1}}{h_n})^m < (\frac{2}{3})^m < \frac{2}{3}$ . Now, since  $m > 1$  and the sequence  $h_n$  is strictly increasing for  $n \geq 3$ , it follows that the sequence  $\frac{1}{(h_n)^{m-1}} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, there exist  $n_0$  such that  $n \geq n_0$  implies  $\frac{h_{n+1}}{(h_n)^m} = \frac{1}{(h_n)^{m-1}} + \left(\frac{h_{n-1}}{h_n}\right)^m \leq \frac{2}{3} + \varepsilon$  for any  $\varepsilon > 0$ . In particular, for large enough  $n$  we have

$$h_{n+1} < (h_{2n})^m.$$

Which can be written as

$$\begin{aligned} \left(\exp\left(\sum_{j=0}^{n-1} \frac{\log \alpha_{2j+1}}{m^{j+1}}\right)\right)^{m^n} &< \left(\left(\exp\left(\sum_{j=0}^{n-1} \frac{\log \alpha_{2j}}{m^{j+1}}\right)\right)^{m^n}\right)^m \\ &= \left(\exp\left(\sum_{j=0}^{n-1} \frac{\log \alpha_{2j}}{m^j}\right)\right)^{m^n}. \end{aligned}$$

Which, since both the exponential and power functions are injective, certainly implies

$$\sum_{j=0}^{n-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} < \sum_{j=0}^{n-1} \frac{\log \alpha_{2j}}{m^j} \tag{8}$$

for all  $n$  sufficiently large. Hence, provided this strict inequality is maintained as  $n \rightarrow \infty$  (which will be shown), there exist  $N$  such that whenever  $n > N$  we

have

$$\sum_{j=0}^{n-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} + \frac{\sqrt{m} \log 3}{m^{n+1}(m-1)} < \sum_{j=0}^{n-1} \frac{\log \alpha_{2j}}{m^j} \quad (9)$$

since  $\frac{\sqrt{m} \log 3}{m^{n+1}(m-1)} \rightarrow 0$  as  $n \rightarrow \infty$  for any  $m$ . Note that conditions (9) and (8) are equivalent. At this point, let us define a set  $E$  as the set of all  $m$  such that condition (9) is satisfied for all real  $x \geq m$ . We will show that  $E$  is nonempty, specifically, that  $3.71 \in E$ . From lemma 2, we found that  $\log \alpha_n \leq \log 3$  for all  $n$  so we may overestimate the left hand side of (9) as follows

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} &\leq \frac{\log \alpha_1}{m} + \log 3 \left( \frac{1}{m^2} + \frac{1}{m^3} + \dots \right) \\ &= \frac{\log 2}{m} + \frac{\log 3}{m^2} \left( \frac{1}{1 - \frac{1}{m}} \right) \\ &= \frac{\log 2}{m} + \frac{\log 3}{m(m-1)}. \end{aligned}$$

Similarly, we make the following underestimate of the right hand side of (9)

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{\log \alpha_{2j}}{m^j} &> \frac{\log \alpha_0}{m^0} + \frac{\log \alpha}{m} \\ &= \log 1 + \frac{\log 3}{m} \\ &= \frac{\log 3}{m}. \end{aligned}$$

Therefore, if it is the case that

$$\frac{\log 2}{m} + \frac{\log 3}{m(m-1)} < \frac{\log 3}{m},$$

then (9) is true. Solving the previous equation for  $m$  shows that whenever

$$m > \frac{\log 3}{\log \frac{3}{2}} + 1 \approx 3.71,$$

then (9) is certainly true. Therefore, it follows that  $3.71 \in E$  and that  $E$  is nonempty. Also, by definition, it is clear that  $E$  is lowerbounded by 1 so that  $\inf E$  exists. Let  $v = \inf E$ . It follows from the previous discussion that  $1 \leq v$ . Suppose  $v > 1$ . Let  $\delta = (v-1)/2$ . Now let us define two sequences of functions

$$f_n, g_n : (1 + \delta, \infty) \rightarrow \mathbb{R}$$

by

$$f_n(m) = \sum_{j=0}^n \frac{\log \alpha_{2j+1}}{m^{j+1}}$$

and

$$g_n(m) = \sum_{j=0}^n \frac{\log \alpha_{2j}}{m^j}$$

By lemma (2) we found that  $f_n$  and  $g_n$  actually make sense and also converge. Say that  $f_n \rightarrow f$  and  $g_n \rightarrow g$ . Lemma (2) also says that  $f_n$  and  $g_n$  are within  $\frac{\log 3}{m^{N+1}(m-1)}$  of  $f$  and  $g$ , respectively, for  $N \leq n$ . It follows then that  $\{f_n\}$  and  $\{g_n\}$  are uniformly convergent on  $(1 + \delta, \infty)$ . Also, by their constructions,  $\frac{\log \alpha_{2j+1}}{m^{j+1}}$  and  $\frac{\log \alpha_{2j}}{m^j}$  are continuous on  $(1 + \delta, \infty)$  for every  $j$ . Therefore, since  $\{f_n\}$  and  $\{g_n\}$  are uniformly convergent series of continuous functions on  $(1 + \delta, \infty)$ , it follows that  $f$  and  $g$  themselves are continuous on  $(1 + \delta, \infty)$ .

Set  $h(m) = g(m) - f(m)$ . It should be clear that  $h(m) > 0$  is equivalent to saying  $m \in E$ . Now, by the definition of  $\inf E$ , we may find a sequence of points  $\{x_n\}$  in  $E$  such that  $x_n \rightarrow v$  and  $h(x_n) > 0$  for every  $n$ . So, by the continuity of  $h$  (which is clearly continuous, being the difference of two continuous functions) it follows that  $h(v) > 0$ . But continuity allows us to go a bit further and pick up  $\epsilon > 0$  such that  $v - \epsilon \in (1 + \delta, \infty)$  and  $h(v - \epsilon) > 0$ . Therefore  $v - \epsilon \in E$  which contradicts the fact that  $v = \inf E$ . Therefore, it follows that  $v = 1$  and that condition (9) holds for all  $m > 1$ .

At this point we define

$$A_* = \exp \left( \sum_{j=0}^{N-1} \frac{\log \alpha_{2j}}{m^{j+1}} \right)$$

and

$$A^* = \exp \left( \sum_{j=0}^{N-1} \frac{\log \alpha_{2j}}{m^{j+1}} + \frac{\log 3}{m^N(m-1)} \right).$$

So, by definition of  $A_m$  and the error estimates obtained in lemma 2, it follows that  $A_* \leq A_m \leq A^*$ . Likewise, we define

$$B_* = \exp \left( \frac{1}{\sqrt{m}} \sum_{j=0}^{N-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} \right)$$

and

$$B^* = \exp \left( \frac{1}{\sqrt{m}} \sum_{j=0}^{N-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} + \frac{\log 3}{m^N(m-1)} \right).$$

Hence, by definition of  $B_m$  and the error estimates obtained, it is clear that  $B_* \leq B_m \leq B^*$ . So, by equation (2), for all  $j \geq N$  we have

$$(A_*)^{\sqrt{m}^{2j}} \leq h_{2j} \leq (A^*)^{\sqrt{m}^{2j}}.$$

Similarly, from equation (4), for all  $j \geq N$  we have

$$(B_*)^{\sqrt{m}^{2j+1}} \leq h_{2j+1} \leq (B^*)^{\sqrt{m}^{2j+1}}.$$

Therefore,

$$\begin{aligned} \frac{h_{2j+1}}{(h_{2j})^m} &\leq \frac{(B^*)^{\sqrt{m}^{2j+1}}}{((A_*)^{\sqrt{m}^{2j}})^m} \\ &= \left( \frac{B^*}{(A_*)^{\sqrt{m}}} \right)^{\sqrt{m}^{j+1}}. \end{aligned}$$

Expanding the argument of the previous expression reveals

$$\begin{aligned} \frac{B^*}{(A_*)^{\sqrt{m}}} &= \frac{\exp\left(\frac{1}{\sqrt{m}} \sum_{j=0}^{N-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} + \frac{\log 3}{m^N(m-1)}\right)}{\exp\left(\sqrt{m} \sum_{j=0}^{N-1} \frac{\log \alpha_{2j}}{m^{j+1}}\right)} \\ &= \exp\left(\frac{1}{\sqrt{m}} \sum_{j=0}^{N-1} \frac{\log \alpha_{2j+1}}{m^{j+1}} + \frac{\log 3}{m^N(m-1)} - \sqrt{m} \sum_{j=0}^{N-1} \frac{\alpha_{2j}}{m^{j+1}}\right). \end{aligned}$$

Dividing (9) by  $\sqrt{m}$  shows that the argument of the previous expression is negative. Whence,

$$0 < \frac{B^*}{(A_*)^{\sqrt{m}}} < 1.$$

Let  $C = \frac{B^*}{(A_*)^{\sqrt{m}}}$ . Then, for  $j \geq N$  we have  $\frac{h_{2j+1}}{(h_{2j})^m} \leq C^{\sqrt{m}^{j+1}}$ . Now since  $m > 1$  we may take  $j$  so large that  $\sqrt{m}^{j+1} > j$ . So, for  $j$  sufficiently large we then have

$$\frac{h_{2j+1}}{(h_{2j})^m} \leq C^{\sqrt{m}^{j+1}} < C^j.$$

Consequently, for  $n$  sufficiently large we then have

$$\begin{aligned} m^n \sum_{j=n}^{\infty} \frac{\log \alpha_{2j}}{m^{j+1}} &\leq m^n \sum_{j=n}^{\infty} \frac{1}{m^{j+1}} C^j \\ &= \frac{1}{m} C^n \left( \frac{1}{1 - \frac{C}{m}} \right). \end{aligned}$$

And since  $0 < C < 1$  this certainly approaches 0 as  $n \rightarrow \infty$ , verifying the first limit statement. The second, for odd integers, follows similarly. ■

## References

- [1] W. Duke, S. Greenfield, E. Speer, *Properties of a quadratic Fibonacci recurrence*, Rutgers University, New Brunswick NJ, date.



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